The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2

Date: April 8, 2009

Course: EE 313 Evans

Name: Set, Solution
Last, First

- The exam is scheduled to last 50 minutes.
- Open books and open notes. You may refer to your homework assignments and the homework solution sets.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network. ***Please disable all wireless connections on your computer system.***
- Please turn off all cell phones, pagers, and personal digital assistants (PDAs).
- All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.**

Problem	Point Value	Your score	Topic
1	20		Difference Equation
2	20		Discrete-Time Convolution
3	40		Transfer Functions
4	20		Potpourri
Total	100		

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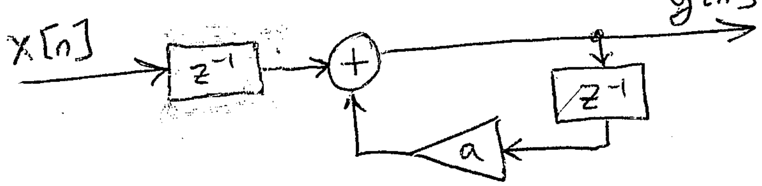
Problem 2.1 Difference Equation. *20 points.*

A causal discrete-time linear time-invariant filter with input $x[n]$ and output $y[n]$ is governed by the following difference equation

$$y[n] = a\,y[n-1] + x[n-1]$$

where $|a| < 1$.

(a) Draw the block diagram for this filter. *4 points.*



z^{-1} represents a delay of one sample (slide 15-4). Lathi's book uses D operator instead of z^{-1} (page 261).

(b) What are the initial conditions? What values should they be assigned and why? *4 points.*

Initial conditions can be found by setting $n=0$ and advancing n:

$y[0] = a\,y[-1] + x[-1]$ ⟹ Initial conditions are $y[-1]$ and $x[-1]$.

$y[1] = a\,y[0] + x[0]$

The initial conditions should be set to zero for the system to be LTI.

(c) What is (are) the values of the characteristic root(s)? *4 points.*

The characteristic polynomial is

$$1 - a\gamma^{-1} = 0$$

which gives one characteristic root at

$$\gamma = a$$

(d) Is the system bounded-input bounded-output stable? Why or why not? *4 points.*

Since $|a| < 1$, the characteristic root is inside the unit circle. Zero-input solution is asymptotically stable (Lathi, p. 314), which implies BIBO stability (Lathi, p. 314).

(e) Let the input $x[n] = \delta[n]$. Compute $y[0]$, $y[1]$ and $y[2]$. Infer a formula for the impulse response. *4 points.*

$y[0] = a\,y[-1] + \delta[-1] = 0$

$y[1] = a\,y[0] + \delta[0] = 1$

$y[2] = a\,y[1] + \delta[1] = a$

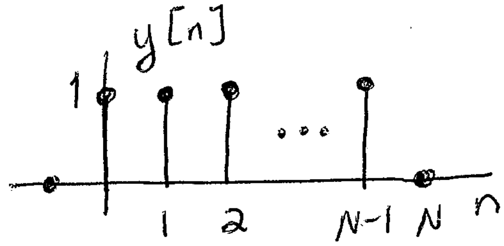
Here, $y[0]$, $y[1]$ and $y[2]$ are first three samples of the impulse response. $h[n] = a^{n-1}\,u[n-1]$

Problem 2.2 Discrete-Time Convolution. *20 points.*

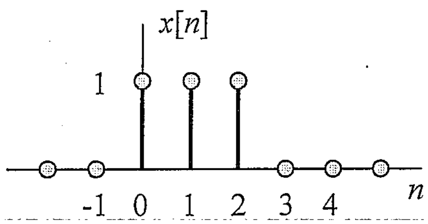
(a) In discrete time, convolve the unit step function $u[n]$ and the function $\delta[n] - \delta[n-N]$, where $\delta[n]$ is the Kronecker impulse and N is a positive integer. *10 points.*

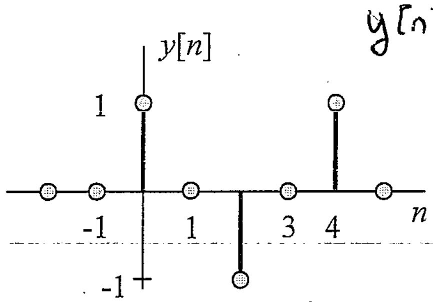
$$y[n] = u[n] * (\delta[n] - \delta[n-N])$$
$$= u[n] * \delta[n] - u[n] * \delta[n-N]$$
$$= u[n] - u[n-N]$$

$y[n]$ is a causal rectangular pulse of length N samples.



(b) Consider a discrete-time linear time-invariant system. For input $x[n]$ given below, the system gives output $y[n]$ below. What is the impulse response of the system? *10 points.*





$$y[n] = h[n] * x[n]$$
$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Since $x[n]$ and $y[n]$ are causal, $h[n]$ must be causal.

Since $x[n]$ and $y[n]$ are finite in extent, assume that $h[n]$ is finite in extent:

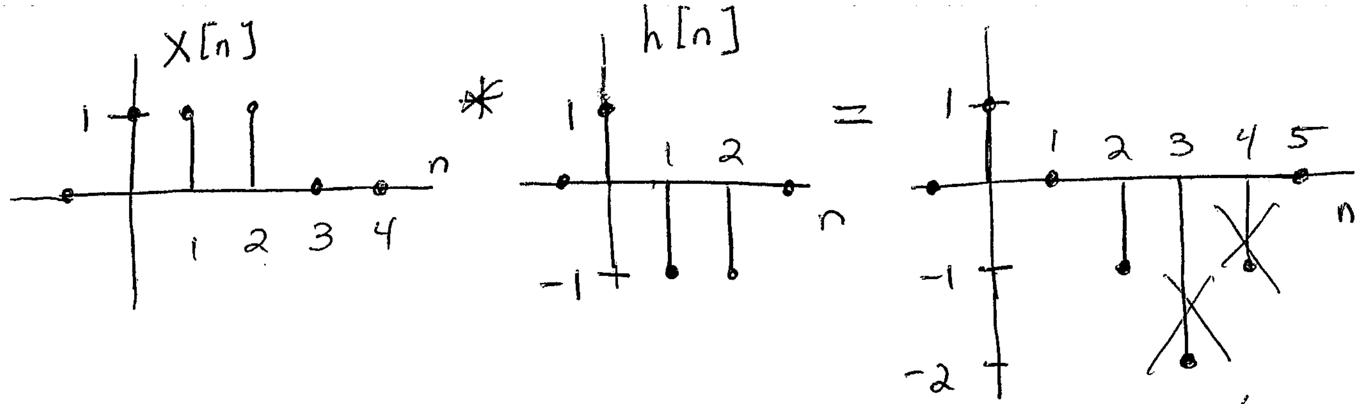
$$L_x + L_h - 1 = L_y \Rightarrow L_h = L_y - L_x + 1$$
$$L_h = 5 - 3 + 1 = 3$$

Assume $h[n] = h[0]\,\delta[n] + h_1\,\delta[n-1] + h_2\,\delta[n-2]$.

$$y[0] = h[0]\,x[0] \Rightarrow h[0] = 1$$
$$y[1] = h[0]\,x[1] + h[1]\,x[0] \Rightarrow h[1] = -1$$
$$y[2] = h[0]\,x[2] + h[1]\,x[1] + h[2]\,x[0]$$
$$-1 = 1 \cdot 1 + (-1) \cdot 1 + h[2] \cdot 1 \Rightarrow h[2] = -1$$

see next page → L-39

2.2.(b) Continued.



The convolution produces the correct first three values of $y[n]$ but is incorrect in the fourth and fifth values.
Let's keep computing additional values of $h[n]$:

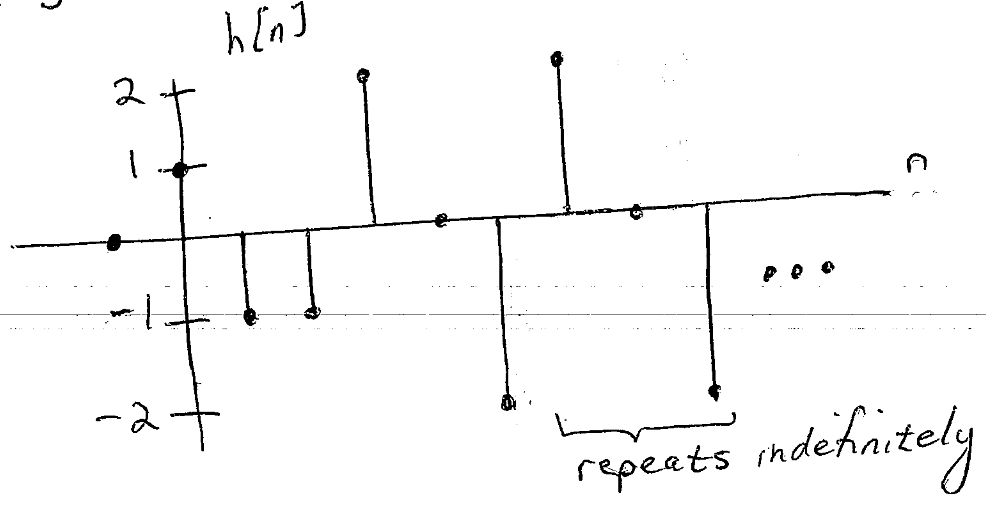
$$y[3] = h[0]x[3] + h[1]x[2] + h[2]x[1] + h[3]x[0] = 0$$
$$= 0 + (-1) + (-1) + h[3] = 0$$

$h[3] = 2$

$$y[4] = h[2]x[2] + h[3]x[1] + h[4]x[0]$$
$$= (-1) + (2) + h[4] = 1$$

$h[4] = 0$
$h[5] = -2$

$h[n]$ is infinite in extent.



Problem 2.3 Transfer Functions. *40 points.*

A causal linear time-invariant (LTI) continuous-time system has the following transfer function in the Laplace transform domain:

$$H(s) = \frac{s}{s+1}$$

(a) Find the corresponding differential equation using $x(t)$ to denote the input signal and $y(t)$ to denote the output signal. Give the minimum number of initial conditions, and their values. *8 points.*

$$H(s) = \frac{s}{s+1} = \frac{Y(s)}{X(s)} \Rightarrow (s+1)Y(s) = s\,X(s) \Rightarrow s\,Y(s) + Y(s) = s\,X(s)$$

Taking the inverse Laplace transform of both sides,

$y'(t) + y(t) = x'(t)$ with initial conditions $y(0^-) = 0$ and $x(0^-) = 0$.

(b) Give the pole location(s) and the region of convergence. *8 points.*

Poles are the roots of the denominator of the transfer function, i.e. $s = -1$.

Region of convergence cannot include a pole location. Due to causality, $\mathrm{Re}\{s\} > -1$

(c) Compute the impulse response by taking the inverse Laplace transform of $H(s)$. *8 points.*

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s+1}\right\}$$

Solution #1: Partial fractions.

$$\mathcal{L}^{-1}\left\{\frac{s}{s+1}\right\} = \mathcal{L}^{-1}\left\{1 - \frac{1}{s+1}\right\} = \delta(t) - e^{-t}u(t)$$

Solution #2: Differentiation Property

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}u(t)$$

$$s\,F(s) \xleftarrow{\mathcal{L}^{-1}} \frac{d}{dt} f(t)$$

$$h(t) = -e^{-t}u(t) + e^{-t}\delta(t)$$

(d) Give a formula for the frequency response. *8 points.*

Since the region of convergence includes the imaginary axis,

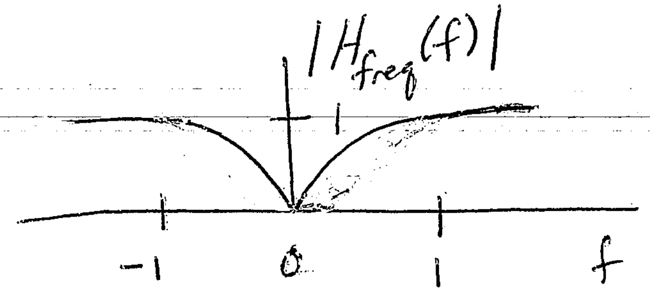
$$H_{freq}(f) = H(s)\Big|_{s=j2\pi f} = \frac{j2\pi f}{j2\pi f + 1}$$

(e) Plot the magnitude of the frequency response and describe the system's frequency selectivity (lowpass, highpass, bandpass, or bandstop). *8 points.*

$$H_{freq}(0) = 0$$

$$\lim_{f\to\infty} \left|H_{freq}(f)\right| = \lim_{f\to\infty} \left|\frac{1}{1 + \frac{1}{j2\pi f}}\right| = 1$$

Highpass filter



L-41

Problem 2.4 Potpourri. 20 points.

(a) *Either prove the following statement to be true, or give a counterexample to show that the following statement is false*: The convolution of two continuous-time signals $x(t)$ and $y(t)$ may always be computed by taking the Laplace transforms of $x(t)$ and $y(t)$, multiplying the Laplace transforms, and applying the inverse Laplace transform to the result.. *5 points.*

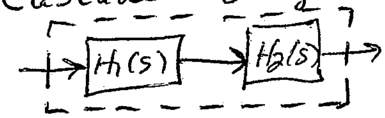
FALSE. Consider $x(t) = e^{t^2} u(t)$, which does not have a Laplace transform. See slide 11-11.

Also see page 405 of Lathi's book, second paragraph from bottom of page.

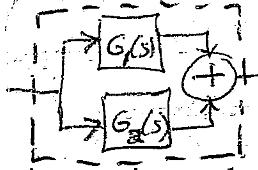
(b) Consider wanting to map a transfer function in the Laplace domain into an implementation. The transfer function is a ratio of two polynomials. When would you recommend using a parallel implementation over a cascaded implementation? *5 points.*

The parallel implementation would have shorter delay through system.

Cascade: Delay is $\text{delay}_1 + \text{delay}_2$.



Delay is the system time constant.



Parallel: Delay is $\max(\text{delay}_1, \text{delay}_2)$.

The parallel implementation is a natural mapping of the result of applying partial fractions decomposition. A parallel form would be natural when numerator order is greater than denominator order.

(c) Give one application that uses a differentiator in continuous time and give one application that uses a differentiator in discrete time. *5 points.*

A differentiator is a highpass filter — see slides 8-9 and 13-13 (in-class comments). Discrete-time — detect edges and texture in images (slide 8-9). Enhance high frequencies in sampled audio (pp. 263-265) in Lathi

Continuous-time — estimate acceleration from a velocity signal. Differential amplifiers.

(d) Given a system governed by a linear constant-coefficient differential equation, the zero-state solution is always bounded-input bounded-output unstable if the zero-input solution is unstable. *5 points.* Either prove the preceding statement to be true, or give a counterexample to show that the preceding statement is false. FALSE.

Consider slide 13-10, which is from example 4.14 in Lathi.

Here is a simpler example based example 4.14:

$y'(t) - y(t) = x'(t) - x(t)$

Zero-input solution is of the form $C_0 e^{t} u(t)$. Unstable.

Zero-state solution is $h(t) * x(t)$ $\quad H(s) = \dfrac{s-1}{s-1} = 1$.

Hence, $h(t) = \delta(t)$. Zero-state solution is simply $x(t)$.

Bounded input gives bounded zero-state solution.